Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **32** (2012) no. 2, 135–140. doi:10.1285/i15900932v32n2p135

A Note on Groups with Just-Infinite Automorphism Groups

Francesco de Giovanni
Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, via Cintia, I - 80126 Napoli (Italy) degiovan@unina.it

Diana Imperatore
Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, via Cintia, I - 80126 Napoli (Italy) diana.imperatore@unina.it

Received: 6.9.2012; accepted: 5.10.2012.

Abstract. An infinite group is said to be *just-infinite* if all its proper homomorphic images are finite. We investigate the structure of groups whose full automorphism group is just-infinite.

Keywords: just-infinite group; automorphism group

MSC 2000 classification: 20E36

1 Introduction

Let D_∞ be the infinite dihedral group. Although D_∞ admits non-inner automorphisms, it is well known that D_∞ is isomorphic to its full automorphism group $Aut(D_\infty)$. Moreover, it has been proved in [3] that there are no other groups with infinite dihedral automorphism group. In other words, the equation

$$Aut(X) \simeq D_\infty$$

admits, up to isomorphisms, only the trivial solution $X = D_\infty$. It is usually hard to understand which groups Q can occur as full automorphism groups of some other group, i.e. when the equation

$$Aut(X) \simeq Q$$

admits at least one solution. For instance, it was proved by D.J.S. Robinson [8] that no infinite Černikov group can be realized as full automorphism group of a group, while a classical result of R. Baer [1] showed that for any finite group Q the above equation has no solution within the universe of infinite periodic groups.

The aim of this paper is to obtain informations about groups whose automorphism groups are just-infinite. Here a group G is said to be *just-infinite* if it is infinite but all its proper homomorphic images are finite. It follows from Zorn's Lemma that any finitely generated infinite group has a just-infinite homomorphic image, and so just-infinite groups play a relevant role in many problems of the theory of infinite groups (see for instance [4]). The first examples of just-infinite groups are of course the infinite cyclic group and the infinite dihedral group; the first of these cannot occur as the full automorphism group of any group, and we discussed above the dihedral case. We will prove that if G is any group admitting an ascending normal series whose factors are either central or finite, then the automorphism group $Aut(G)$ cannot be just-infinite. It follows in particular that in any group with just-infinite automorphism group, the centre and the hypercentre coincide. Some examples of groups with just-infinite automorphism groups will also be constructed.

Most of our notation is standard and can for instance be found in [7].

2 Results and examples

The structure of just-infinite groups has been described by J.S. Wilson [11]; of course, all infinite simple groups are just-infinite, while any soluble-by-finite just-infinite group is a finite extension of a free abelian group of finite rank. In the same paper, the class $\mathfrak{D}_2$, consisting of all infinite groups in which every non-trivial subnormal subgroup has finite index, is considered; obviously, all $\mathfrak{D}_2$-groups are just-infinite, but it is clear that any $\mathfrak{D}_2$-group containing an abelian non-trivial subnormal subgroup is either cyclic or dihedral. It follows easily from this remark that the result proved in [3] can be extended to the next statement (recall here that a group is called *generalized subsoluble* if it has an ascending subnormal series whose factors are either abelian or finite).

Theorem 1. *Let G be a generalized subsoluble group whose automorphism group $Aut(G)$ is a $\mathfrak{D}_2$-group. Then $G \simeq D_\infty$.*

The following example shows that a similar result cannot be proved for groups with just-infinite automorphism groups, even restricting the attention to the case of polycyclic groups.

Let $A = \langle a \rangle \times \langle b \rangle$ be a free abelian group of rank 2, and let x and y be the automorphisms of A defined by the positions

$$a^x = b, \quad b^x = a, \quad a^y = b, \quad b^y = a^{-1}b.$$

Then $\langle x, y \rangle$ is a dihedral subgroup of order 12 of $GL(2, \mathbb{Z})$, and the semidirect product

$$G = \langle x, y \rangle \ltimes A$$

is a polycyclic group. Moreover, A is self-centralizing and has no cyclic non-trivial G-invariant subgroups, so that G is just-infinite. On the other hand, the group G is complete, i.e. it has trivial centre and $Aut(G) = Inn(G)$, and hence $Aut(G) \simeq G$ (see [9]).

Lemma 1. *Let G be an abelian group. If all proper homomorphic images of the full automorphism group $Aut(G)$ of G are finite, then $Aut(G)$ is finite.*

Proof. Assume for a contradiction that $Aut(G)$ is just-infinite. As the inversion map τ of G belongs to the centre $Z(Aut(G))$, we have that $\langle \tau \rangle$ is a finite normal subgroup of $Aut(G)$, so that τ is the identity and G is an infinite abelian group of exponent 2. Let Γ be the set of all automorphisms α of G acting trivially on a subgroup of finite index of G. Then Γ is a non-trivial normal subgroup of $Aut(G)$ and the index $|Aut(G) : \Gamma|$ is infinite. This contradiction proves the lemma. $\boxed{QED}$

Lemma 2. *Let G be a just-infinite group, and let N be a normal subgroup of G. Then N has no finite non-trivial normal subgroups.*

Proof. Let X be any finite normal subgroup of N. Since G/N is finite, the conjugacy class of X in G is finite, and so it follows from the well known Dietzmann's lemma that the normal closure X^G is a finite normal subgroup of G. Therefore $X = \{1\}$ and the lemma is proved. $\boxed{QED}$

We can now prove our main result on groups with just-infinite automorphism group.

Theorem 2. *Let G be a group admitting an ascending normal series whose factors are either central or finite. Then the automorphism group $Aut(G)$ is not just-infinite.*

Proof. Assume for a contradiction that the group $Aut(G)$ is just-infinite, and let

$$\{1\} = G_0 < G_1 < \ldots < G_\alpha < \ldots < G_\tau = G$$

be an ascending normal series whose infinite factors are central. As the inner automorphism group $Inn(G)$ is a non-trivial normal subgroup of $Aut\,G$, it follows from Lemma 2 that $Inn(G)$ has no finite non-trivial normal subgroups. Then the centre $Z(Inn(G))$ is non-trivial, and so it has finite index in $Aut(G)$. In particular, the index $|G : Z_2(G)|$ is finite, and hence the term $\gamma_3(G)$ of the lower central series of G is finite (see [7] Part 1, p.113). Thus $\gamma_3(G)$ is contained in $Z(G)$, and G is nilpotent, so that $Inn(G)$ lies in the Fitting subgroup of $Aut(G)$. Therefore

$$G/Z(G) \simeq Inn(G)$$

is a free abelian group of finite rank r (see [11], Theorem 2). It is well known that the homomorphism group

$$Hom\big(G/Z(G), Z(G)\big)$$

is isomorphic to an abelian normal subgroup of $Aut(G)$, so that it is torsion-free and hence also $Z(G)$ must be torsion-free; moreover, $Hom\big(G/Z(G), Z(G)\big)$ is isomorphic to the direct product of r copies of $Z(G)$. On the other hand, the groups $Inn(G)$ and $Hom\big(G/Z(G), Z(G)\big)$ are isomorphic, and hence $Z(G)$ is infinite cyclic and G is torsion-free.

Put $C = Z(G)$ and $Q = G/C$, and let x be any element of $G \setminus C$. The mapping

$$\varphi : g \longmapsto [g, x]$$

is a non-trivial homomorphism of G into C, whose kernel coincides with the centralizer $C_G(x)$, so that $G/C_G(x)$ is infinite cyclic and

$$G = \langle y \rangle \ltimes C_G(x)$$

for some element y of infinite order. Let m be a non-negative integer such that $(yx)^m$ belongs to $C_G(x)$. As $(yx)^m = y^m x^m z$ for some $z \in C_G(x)$, we have that y^m is in $C_G(x)$, and so $m = 0$. Therefore $\langle yx \rangle \cap C_G(x) = \{1\}$, and hence

$$G = \langle yx \rangle \ltimes C_G(x).$$

It follows that an automorphism α of G can be defined by setting

$$y\alpha = yx \quad \text{and} \quad c\alpha = c$$

for all $c \in C_G(x)$. Then $y\alpha^n = yx^n$ for each positive integer n, so that α has infinite order and α^n cannot be an inner automorphism of G. This is of course a contradiction, since $Inn(G)$ has finite index in $Aut(G)$. $\boxed{QED}$

The above theorem shows in particular that hypercentral groups cannot have just-infinite automorphism groups. We leave here as an open question whether there exists a locally nilpotent group whose automorphism group is just-infinite. As a consequence of Theorem 2, we can observe that the upper central series of any group with just-infinite automorphism group stops at the centre.

Corollary 1. *Let G be a group whose automorphism group $Aut(G)$ is just-infinite. Then $Z(G) = Z_2(G)$.*

Proof. As $Aut(G)$ is just-infinite, it follows from Theorem 2 that the index $|G : Z_2(G)|$ must be infinite, and hence $Z(G) = Z_2(G)$ because $Z_2(G)$ is a characteristic subgroup of G. $\boxed{QED}$

As infinite simple groups are just-infinite, we have that complete infinite simple groups are trivial examples of groups with just-infinite automorphism groups. Among such groups we find for instance the universal locally finite groups of cardinality $2^{\aleph_0}$ (see [5]); recall here that a locally finite group U is said to be *universal* if it contains a copy of any finite group and any two finite isomorphic subgroups of U are conjugate.

Our last result shows how to find examples of non-simple $\mathfrak{D}_2$-groups occurring as full automorphism groups; here the main ingredient is an infinite simple group with finite non-trivial outer automorphism group. Groups of this kind have for instance been constructed by R.J. Thompson in his study of homeomorphisms of the circle $S^1 = \mathbb{R}/\mathbb{Z}$ (see [2]). Recall here that the *outer automorphism group* of a group G is the factor group $Out(G) = Aut(G)/Inn(G)$.

Lemma 3. *Let G be a group containing a simple normal subgroup N such that G/N is finite and $C_G(N) = \{1\}$. Then every non-trivial subnormal subgroup of G has finite index.*

Proof. Let S be any non-trivial subnormal subgroup of G. Then $[N, S]$ cannot be trivial, and hence $S \cap N \neq \{1\}$ (see for instance [10], 13.3.1). Thus S contains N, and so the index $|G : S|$ is finite. $\boxed{QED}$

Theorem 3. *Let G be an infinite simple group whose outer automorphism group $Out(G)$ is finite. Then $Aut(G)$ is an infinite complete group whose non-trivial subnormal subgroups have finite index.*

Proof. The automorphism group $Aut(G)$ is complete by a well known result of Burnside (see for instance [10], 13.5.9), and $Inn(G) \simeq G$ is a simple normal subgroup of $Aut(G)$ of finite index. Moreover, as $Z(G) = \{1\}$, also the centralizer $C_{Aut(G)}(Inn(G))$ is trivial, and hence it follows from Lemma 3 that any non-trivial subnormal subgroup of $Aut(G)$ has finite index. $\boxed{QED}$

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